FLOW OF WATER INTO A HORIZONTAL TUBULAR DRAIN IN A PRESSURIZED TWO-LAYER STRATUM OF LIMITED THICKNESS

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Liu [1] has discussed the particular case of flow into a tubular drain in a two-layer stratum of limited thickness, the top of the upper layer being an equipotential (bottom of a reservoir). Here we consider the particular case where the influx area has a rectangular edge (Fig. 1).

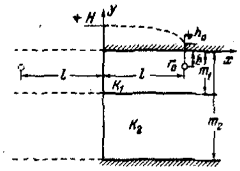

Fig. 1

At point (l, b) in the upper layer we place the drain (sink), the flow rate being q. To satisfy the boundary condition at the source for x = 0 we place to the left of the axis a symmetrical source of equal flow rate. The final solution is obtained by summing the solutions for sink and source.

The complex velocity in the upper layer is

$$w_1 = -\frac{q}{2\pi k_1}\left[\frac{1}{z-\zeta}+\frac{1}{z-\bar{\zeta}}\right]+$$
$$+\int_0^\infty [A_1(\alpha)e^{i\alpha z}+B_1(\alpha)e^{-i\alpha z}]\,d\alpha$$
$$(\zeta = l - ib,\quad \bar{\zeta} = l + ib) \tag{1}$$

while that in the lower layer is

$$w_2 = \int_0^\infty [A_2(\alpha)e^{i\alpha z}+B_2(\alpha)e^{-i\alpha z}]\,d\alpha. \tag{2}$$

The quantities $A_1(\alpha)$, $A_2(\alpha)$, $B_1(\alpha)$, $B_2(\alpha)$ of (1) and (2) are complex functions of the real variable α.

The vertical velocity must be zero at the impermeable boundaries of the two layers,

$$\mathrm{Im}\,(w_1) = 0 \ \text{ for } y = 0, \ \mathrm{Im}\,(w_2) = 0 \ \text{ for } y = -m_2,$$

so from (1) and (2) we get

$$B_1(\alpha) = \bar{A}_1(\alpha),\ B_2(\alpha) = [\bar{A}_2(\alpha)e^{2m_2\alpha}].$$

To be able to use the conditions at the boundary between the layers, we represent the main part of the expression for w_1 as a definite integral:

for $\mathrm{Im}\,(z-\zeta)<0$

$$\frac{1}{z-\zeta} = i\int_0^\infty e^{-i\alpha(z-\zeta)}\,d\alpha = i\int_0^\infty e^{-i\alpha(z-l+ib)}d\alpha,$$

for $\mathrm{Im}\,(z-\bar{\zeta})<0$

$$\frac{1}{z-\bar{\zeta}} = i\int_0^\infty e^{-i\alpha(z-l-ib)}\,d\alpha.$$

The above transformations give the expressions for the complex velocities as

$$w_1(z) = \int_0^\infty \left[\frac{iq}{\pi k_1}\,\mathrm{ch}\,\alpha b e^{-i\alpha(z-l)} + A_1 e^{i\alpha z} + \bar{A}_1 e^{-i\alpha z}\right] d\alpha, \tag{3}$$

$$w_2(z) = \int_0^\infty [A_2 e^{i\alpha z} + \bar{A}_2 e^{2\alpha m_2 - i\alpha z}]\,d\alpha. \tag{4}$$

The normal component of the velocity must be continuous at the interface, while the tangential components are proportional to the filtration coefficients

$$\mathrm{Im}\,(w_2) = \mathrm{Im}\,(w_1),\quad k_1\,\mathrm{Re}\,(w_2) = k_2\,\mathrm{Re}\,(w_1)\quad \text{for } y = -m_1.$$

From (3), after satisfying the conditions at the interface, we get

$$A_1(\alpha) =$$
$$= \frac{q}{\pi k_1}\,\mathrm{ch}\,\alpha b\,\frac{1+\lambda e^{2\alpha(m_2-m_1)}}{1-\lambda e^{2\alpha m_1}+\lambda e^{2\alpha(m_2-m_1)}-e^{2\alpha m_2}}\,(\sin\alpha l + i\cos\alpha l), \tag{5}$$

$$\bar{A}_1(\alpha) = \frac{q}{\pi k_1}\,\mathrm{ch}\,\alpha b\,\frac{1+\lambda e^{2\alpha(m_2-m_1)}}{1-\lambda e^{2\alpha m_1}+\lambda e^{2\alpha(m_2-m_1)}-e^{2\alpha m_2}}\,\cdot$$
$$\cdot(\sin\alpha l - i\cos\alpha l)\quad \left(\lambda = \frac{k_1-k_2}{k_1+k_2}\right). \tag{6}$$

In (5) and (6) we make the substitutions

$$\sin\alpha l + i\cos\alpha l = ie^{-i\alpha l},\qquad \sin\alpha l - i\cos\alpha l = -ie^{i\alpha l}.$$

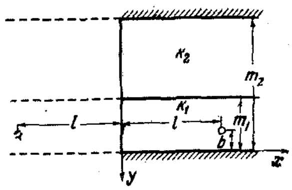

Fig. 2

The expression for the complex velocity in the upper region becomes

$$w_1(z) = \int_0^\infty \frac{iq}{\pi k_1}\,\mathrm{ch}\,\alpha b\,[e^{i\alpha(l-z)} + 2N_1\,\mathrm{sh}\,i\alpha(z-l)]\,d\alpha, \tag{7}$$

$$N_1(\alpha) = \frac{1+\lambda e^{2\alpha(m_2-m_1)}}{1-\lambda e^{2\alpha m_1}+\lambda e^{2\alpha(m_2-m_1)}-e^{2\alpha m_2}}. \tag{8}$$

Expression (8) may [1] be put as

$$N_1(\alpha) = -\frac{s^{p_2}+\lambda s^{p_1}}{1-\lambda s^{p_1}+\lambda s^{p_2-p_1}-s^{p_2}} \equiv \sum_{n=1} c_n s^n = \sum c_n e^{-2\alpha n m_0},$$

in which m_0 is the largest common factor of m_1 and m_2,

$$p_1 = \frac{m_1}{m_0}, \quad p_2 = \frac{m_2}{m_0} \ (p_1, p_2 \text{ are integers}),$$

$$s = e^{-2\alpha m_0} < 1,$$

so series $N_1(\alpha)$ converges. We put

$$N_1(\alpha) = \sum_{n=1} c_n e^{-2\alpha n m_0}$$

in (7) to get

$$w_1(z) = \int_0^\infty \frac{iq}{2\pi k_1} \{[e^{-i\alpha(z-l+ib)} + e^{-i\alpha(z-l-ib)}] +$$

$$+ \sum c_n [e^{-i\alpha[l-z-(2nm_0-b)i]} + e^{-i\alpha[l-z-(2nm_0+b)i]} -$$

$$- e^{-i\alpha[z-l-(2nm_0-b)i]} - e^{-i\alpha[z-l-(2nm_0+b)i]}]\}\, d\alpha =$$

$$= \frac{q}{2\pi k_1}\left[\frac{1}{z-l+ib} + \frac{1}{z-l-ib}\right] -$$

$$- \frac{q}{2\pi k_1} \sum c_n \left[\frac{1}{z-l+(2nm_0-b)i} + \right.$$

$$+ \frac{1}{z-l+(2nm_0+b)i} + \frac{1}{z-l-(2nm_0-b)i} +$$

$$\left. + \frac{1}{z-l-(2nm_0+b)i}\right]. \qquad (9)$$

Equation (9) has to be integrated to give the complex potential,

$$W_1(z) = \varphi_1 + i\psi_1 = h_1 + i\psi =$$

$$= \frac{q}{2\pi k_1}[\ln(z-l+ib) + \ln(z-l-ib)] -$$

$$- \frac{q}{2\pi k_1} \sum c_n \{\ln[z-l+(2nm_0-b)i] + \ln[z-l+(2nm_0+b)i] + \qquad (10)$$

$$+ \ln[z-l-(2nm_0-b)i] + \ln[z-l-(2nm_0+b)i]\} + W_0,$$

in which W_0 is a complex quantity.

Separating the real parts, we get the head due to the flow rate q in the upper layer at $(l, -ib)$,

$$h_c = \operatorname{Re} W_1^c(z) =$$

$$= \frac{q}{4\pi k_1}\{\ln[(x-l)^2 + (y+b)^2] + \ln[(x-l)^2 + (y-b)^2]\} -$$

$$- \frac{q}{4\pi k_1} \sum c_n \{\ln[(x-l)^2 + (y+2nm_0-b)^2] +$$

$$+ \ln[(x-l)^2 + (y+2nm_0+b)^2] +$$

$$+ \ln[(x-l)^2 + (y-2nm_0+b)^2] +$$

$$+ \ln[(x-l)^2 + (y-2nm_0-b)^2]\} + C_c. \qquad (11)$$

Similarly we get the head due to the source flow rate $-q$ placed in the upper layer at $(-l, -ib)$:

$$h_u = \operatorname{Re} W_1^u(z) =$$

$$= -\frac{q}{4\pi k_1}\{\ln[(x+l)^2 + (y+b)^2] + \ln[(x+l)^2 + (y-b)^2]\} +$$

$$+ \frac{q}{4\pi k_1} \sum c_n \{\ln[(x+l)^2 + (y+2nm_0-b)^2] +$$

$$+ \ln[(x+l)^2 + (y+2nm_0+b)^2] +$$

$$+ \ln[(x+l)^2 + (y-2nm_0+b)^2] +$$

$$+ \ln[(x+l)^2 + (y-2nm_0-b)^2]\} + C_u. \qquad (12)$$

Summation of (11) and (12) gives the final solution,

$$h = h_c + h_u =$$

$$= \frac{q}{4\pi k_1}\left\{\ln\frac{[(x-l)^2 + (y+b)^2][(x-l)^2 + (y-b)^2]}{[(x+l)^2 + (y+b)^2][(x+b)^2 + (y-b)^2]} - \right.$$

$$- \sum_1 c_n \times$$

$$\times \ln\frac{[(x-l)^2 + (y+2nm_0-b)^2][(x-l)^2 + (y+2nm_0+b)^2]}{[(x+l)^2 + (y+2nm_0-b)^2][(x+l)^2 + (y+2nm_0+b)^2]} \times$$

$$\left. \times \frac{[(x-l)^2 + (y-2nm_0+b)^2][(x-l)^2 + (y-2nm_0-b)^2]}{[(x+l)^2 + (y-2nm_0+b)^2][(x+l)^2 + (y-2nm_0-b)^2]}\right\} + C \qquad (13)$$

To satisfy the condition h = H at x = 0 we must put C = H.

Let the head at the edge of the tubular drain (radius r_0) be h_0. Then, putting $x = l - r_0$ and $y = -b$ in the equation, we have

$$h_0 = \frac{q}{4\pi k_1}\left\{\ln\frac{r_0^2(r_0^2 + 4b^2)}{(2l-r_0)^2[(2l-r_0)^2 + 4b^2]} - \right.$$

$$- \sum_1 c_n \ln\frac{[r_0^2 + 4(nm_0-b)^2]}{[(2l-r_0)^2 + 4(nm_0-b)^2]} \times$$

$$\left. \times \frac{(r_0^2 + 4n^2m_0^2)^2[r_0^2 + 4(nm_0+b)^2]}{[(2l-r_0)^2 + 4n^2m_0^2]^2[(2l-r_0)^2 + 4(nm_0+b)^2]}\right\} + H. \qquad (14)$$

From (14), with $r_0 < m_0$ and $r_0 \ll l$, we get the flow rate q per unit length of drain as

$$q = 4\pi k_1 (H - h_0)\left\{\ln\frac{16l^2(l^2+b^2)}{r_0^2(r_0^2+4b^2)} - \right.$$

$$- \sum_1 c_n \ln\left(1 + \frac{l^2}{(nm_0-b)^2}\right) + 2\ln\left(1 + \frac{l^2}{n^2m_0^2}\right) + \qquad (15)$$

$$\left. + \ln\left(1 + \frac{l^2}{(nm_0+b)^2}\right)\right\}^{-1}.$$

The drain lies at the top of the upper layer when b = 0, and then the formula for the flow rate is

$$q = \pi k_1 (H - h_0)\left[\ln\frac{2l}{r_0} - \sum_1 c_n \ln\left(1 + \frac{l^2}{n^2m_0^2}\right)\right]^{-1}. \qquad (16)$$

If the drain lies in the lower layer, the above relationships are used, but with the disposition as in Fig. 2, which is the mirror image of the previous scheme.

The following are particular cases. If $m_2 \to \infty$ (infinitely thick lower layer),

$$N_1(\alpha) = -\frac{\lambda s^{p_1}}{1 - \lambda s^{p_1}} =$$

$$= -(\lambda s^{p_1} + \lambda^2 s^{2p_1} + \lambda^3 s^{3p_1} + \ldots) = -\sum_1^\infty \lambda^n e^{-2n\alpha m_1},$$

$$q = 4\pi k_1 (H - h_0)\left\{\ln\frac{16l^2(l^2+b^2)}{r_0^2(r_0^2+4b^2)} + \sum_{n=1}^\infty \lambda^n \left[\ln\left(1 + \frac{l^2}{(nm_1-b)^2}\right) + \right.\right.$$

$$\left.\left. + 2\ln\left(1 + \frac{l^2}{n^2m_1^2}\right) + \ln\left(1 + \frac{l^2}{(nm_1+b)^2}\right)\right]\right\}^{-1}, \qquad (17)$$

$$q = \pi k_1 (H - h_0) \left[\ln \frac{2l}{r_0} + \sum_{n=1}^{\infty} \lambda^n \ln \left(1 + \frac{l^2}{n^2 m_1^2} \right) \right]^{-1}. \quad (18)$$

If $p_1 = 1$ and $p_2 = 2$ (two layers equal in thickness), $c_n = -1$ ($n = 2, 4, 6 \ldots$); $c_n = -\lambda$($n = 1, 3, 5 \ldots$); then the formulas for the flow rates may be put as

$$q = 2\pi (k_1 + k_2)(H - h_0) \left[\frac{2\pi l}{m_0} + \frac{k_1 + k_2}{k_1} \ln \frac{m_0}{2\pi r_0} + \frac{k_1 - k_2}{2k_1} \ln \left(\frac{2}{1 - \cos 2\pi b / m_0} \right) + \frac{k_2}{k_1} \ln \frac{(1 - \cos \pi b / m_0)}{8} \right]^{-1}, \quad (19)$$

$$q = \pi (k_1 + k_2)(H - h_0) \left(\frac{\pi l}{m_0} + \frac{k_1 + k_2}{k_1} \ln \frac{4m_0}{\pi r_0} - 2 \ln 2 \right)^{-1}. \quad (20)$$

For $k_2 = 0$, $\lambda = 1$ we get from (17) and (18) or (19) and (20) the known relations for the specific flow rate of a tubular drain in a homogeneous pressurized stratum,

$$q = 2\pi k_1 (H - h_0) \left[\frac{2\pi l}{m_1} + \ln \frac{m_1}{2\pi r_0} + \frac{1}{2} \ln \left(\frac{2}{1 - \cos \frac{2\pi b}{m_1}} \right) \right]^{-1},$$

$$q = \pi k_1 (H - h_0) \left[\frac{\pi l}{m_1} + \ln \frac{m_1}{\pi r_0} \right]^{-1}.$$

Formula (20) also gives Shestakov's approximate formula (see Dissertation, Moscow, VNIIG, 1963), which he derived for the case $k_2/k_1 > 5$–10. The following are the expressions for the first ten coefficients c_n for two cases:

a) $p_1 = 1$, $p_2 = 3$;

$$c_1 = -\lambda$$
$$c_2 = -\lambda^2$$
$$c_3 = -(1 - \lambda^2 + \lambda^3)$$
$$c_4 = -(2\lambda - 2\lambda^3 + \lambda^4)$$
$$c_5 = -(-\lambda + 3\lambda^2 + \lambda^3 - 3\lambda^4 + \lambda^5)$$
$$c_6 = -(1 - 4\lambda^2 + 4\lambda^3 + 3\lambda^4 - 4\lambda^5 + \lambda^6)$$
$$c_7 = -(3\lambda + \lambda^2 - 9\lambda^3 + 4\lambda^4 + 6\lambda^5 - 5\lambda^6 + \lambda^7)$$
$$c_8 = -(-2\lambda + 6\lambda^2 + 6\lambda^3 - 16\lambda^4 + 2\lambda^5 + 10\lambda^6 - 6\lambda^7 + \lambda^8)$$
$$c_9 = -(1 - 9\lambda^2 + 9\lambda^3 + 18\lambda^4 - 24\lambda^5 - 3\lambda^6 + 15\lambda^7 - 7\lambda^8 + \lambda^9)$$
$$c_{10} = -(4\lambda + 3\lambda^2 - 24\lambda^3 + 7\lambda^4 + 40\lambda^5 - 31\lambda^6 - 12\lambda^7 + 21\lambda^8 - 8\lambda^9 + \lambda^{10}),$$

b) $p_1 = 1$, $p_2 = 4$.

$$c_1 = -\lambda$$
$$c_2 = -\lambda^2$$
$$c_3 = -\lambda^3$$
$$c_4 = -(1 - \lambda^2 + \lambda^4)$$
$$c_5 = -(2\lambda - 2\lambda^3 + \lambda^5)$$
$$c_6 = -(3\lambda^2 - 3\lambda^4 + \lambda^6)$$
$$c_7 = -(-\lambda + 5\lambda^3 - 5\lambda^5 + \lambda^7)$$
$$c_8 = -(1 - 4\lambda^2 + 8\lambda^4 - 5\lambda^6 + \lambda^8)$$
$$c_9 = -(3\lambda - 9\lambda^3 + 12\lambda^5 - 6\lambda^7 + \lambda^9)$$
$$c_{10} = -(7\lambda^2 - 17\lambda^4 + 17\lambda^6 - 7\lambda^8 + \lambda^{10}).$$

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